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Darboux transforms, deep reductions and solitons

S B Leble and N V Ustinov

Kaliningrad State University, Theoretical Physics Department, Russia, 236041, Kaliningrad, Al. Nevsky St., 14

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Abstract. Explicit formalisms for deep reductions of matrix differential equations and Darboux covariance properties are presented. The matrix 2×2 spectral problem of the second order is considered. This problem with the reduction constraints being imposed on the potentials is the first equation of the Lax representation of the Hirota–Satsuma system. The two reduction cases are treated with the help of the bilinear δ-forms. The covariance of these forms with respect to the Darboux transforms under restrictions gives rise to explicit formulas of N-soliton solutions. In particular the two-parameter soliton solutions of the Hirota–Satsuma system are obtained. The specific feature of such solutions evolution is that the singularity appears in some parameter region. The Yajima–Oikawa system is given as an example of the technique application to a 3×3 spectral problem.

1. Introduction

Investigations of matrix spectral problems and nonlinear evolution equations that are generated by associated Lax pairs are strongly connected with the problem of reduction restrictions on potentials [1]. In the present paper the formalism of deep reductions with the Darboux transformation (DT) technique [2] is developed. The application to systems of nonlinear evolution equations that appear as a result of reduction constraints in a 2×2 spectral problem of the second order and compatible with the time evolution is presented.

A special case of this reduction is the Hirota–Satsuma (HS) system that has been introduced in [3] where one- and two-soliton solutions have been built within the single-parameter family by means of the Hirota method. The Sato theory has been used to obtain the N-soliton solutions in this one-parameter and the KdV soliton families [4]. For this purpose additional constraints were imposed for solutions of the four-reduction of the KP hierarchy to be solutions of the HS system without investigating possibilities of other constraints, and hence other solutions of the HS system.

Lax pairs [4, 5] and infinitely many conservation laws [4] were found. Bäcklund transformation of this system was used in [6] to rederive the simple soliton solution. The practical realization of this technique failed to address the higher Bäcklund transformation construction. The ψ functions of the associated linear problem and their transformation formulas, which are a specific feature of the Darboux approach, allow one to overcome these difficulties and obtain factorization in transformation formulas in the case of the HS system and in similar systems.

The present paper is organized in the following way. Section 2 is devoted to the covariance of the above-mentioned spectral problem with respect to the standard DT as well as to two elementary DTs that factorize the standard one. The formulas for iteration of the DT, and the conditions keeping the reductions defined by two kinds of automorphisms are presented. In the third section we introduce the bilinear δ-forms that impose additional constraints in the spectral problem having both automorphisms and give DTs which provide that the new potentials possess the same restrictions. The Lax pair, its covariance properties and nonlinear equations are considered and the new two-parameter solitons family for the HS system are built in section 4. As a representative example the application of this technique to the sonic-Langmuir interaction (Yajima–Oikawa) system is given in section 5.

2. The matrix spectral equation of second order and its Darboux covariance

Let us introduce the differential polynomial of second order with spectral parameter λ and 2×2 matrix coefficients

$$\psi_{xx}+F\psi_x+U\psi=\lambda\sigma_3\psi \tag{2.1}$$

where $\sigma_3=\mathrm{diag}(1,-1)$ is the Pauli matrix, $\psi=(\psi_1,\psi_2)^T$ and potentials are $U=\{u_{ij}\}$, $F=(f_{ij}, f_{ii}=0\}$, $i=1,2$, $j=1,2$.

The Matveev theorem [2] deals with the transform of ψ

$$\tilde{\psi}=\psi_x+\varepsilon\psi \qquad \varepsilon=-\phi_x\phi^{-1} \tag{2.2a}$$

where $\phi=(\phi^{(1)},\phi^{(2)})$, $\phi^{(1)}$ and $\phi^{(2)}$ are column solutions of (2.1) with spectral parameters $\mu^{(1)}$ and $\mu^{(2)}$, $\phi^{(i)}=(\varphi_1^{(i)},\varphi_2^{(i)})^T$ $(i=1,2)$ and states the covariance of (2.1) with respect to the DT equation (2.2a) with new (transformed) potentials:

$$\tilde{F}=F+\sigma_3\varepsilon\sigma_3-\varepsilon \qquad \tilde{U}=U+F_x-2\varepsilon_x+\sigma_3\varepsilon\sigma_3F-\tilde{F}\varepsilon. \tag{2.2b}$$

The formulas for transformed $\varphi^{(i)}$ are given by the equation

$$\tilde{\varphi}^{(i)}=(\partial_x+\varepsilon)\partial\varphi^{(i)}/\partial\mu^{(i)}. \tag{2.2c}$$

The Matveev transform itself can be decomposed into two new (elementary) transforms [7].

Lemma 2.1. Equation (2.1) admits two Darboux transforms. The first one is:

$$\tilde{\psi}_1=\psi_{1x}+\varepsilon_{11}\psi_1+\varepsilon_{12}\psi_2 \qquad \varepsilon_{11}=-(\varphi_{1x}+\tfrac{1}{2}f_{12}\varphi_2)\varphi_1^{-1} \qquad \varepsilon_{12}=f_{12}/2$$

$$\tilde{\psi}_2=\psi_2+\varepsilon_{21}\psi_1 \qquad \varepsilon_{21}=-\varphi_2\varphi_1^{-1}$$

$$\tilde{\varphi}_1=(\partial_x+\varepsilon_{11})\varphi_{1,\mu}+\varepsilon_{12}\varphi_{2,\mu} \qquad \tilde{\varphi}_2=\varphi_{2,\mu}+\varepsilon_{21}\varphi_{1,\mu}$$

$$\tilde{f}_{12}=u_{12}+f_{12}\varepsilon_{11} \qquad \tilde{f}_{21}=-2\varepsilon_{21}$$

$$\tilde{u}_{11}=u_{11}-2\varepsilon_{11x}-\tilde{f}_{12}\varepsilon_{21}-f_{21}\varepsilon_{12}$$

$$\tilde{u}_{12}=u_{12x}-\varepsilon_{12xx}+\varepsilon_{11}u_{12}-\varepsilon_{12}(\tilde{u}_{11}+u_{22})$$

$$\tilde{u}_{21}=f_{21}-2\varepsilon_{21x}-\tilde{f}_{21}\varepsilon_{11}$$

$$\tilde{u}_{22}=u_{22}-\varepsilon_{21}u_{12}-\tilde{u}_{21}\varepsilon_{21}-\tilde{f}_{21}\varepsilon_{12x}$$

where $\varphi=(\varphi_1, \varphi_2)^T$ is a solution of (2.1) with spectral parameter μ. The second transform is obtained by means of the indices rearrangement $1\to 2$ and $2\to 1$.

The transforms commute and their product yields the standard Darboux transformation equation (2.2).

The iterations of the DT given by (2.2) give rise to the following formulas for transformed wavefunction and potentials

$$\tilde{\psi}_i[N]=\begin{vmatrix} \psi_{i,Nx} & \psi_{1,(N-1)x} & \psi_{2,(N-1)x} & \cdots & \psi_1 & \psi_2 \\ \varphi^{(1)}_{i,Nx} & \varphi^{(1)}_{1,(N-1)x} & \varphi^{(1)}_{2,(N-1)x} & \cdots & \varphi^{(1)}_1 & \varphi^{(1)}_1 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ \varphi^{(2N)}_{i,Nx} & \varphi^{(2N)}_{1,(N-1)x} & \varphi^{(2N)}_{2,(N-1)x} & \cdots & \varphi^{(2N)}_1 & \varphi^{(2N)}_2 \end{vmatrix} \Big/ \Delta[N]$$

$$= \psi_{i,Nx}+\varepsilon^{(N-1)}_{i1}\psi_{1,(N-1)x}+\varepsilon^{(N-1)}_{i2}\psi_{2,(N-1)x}+\cdots+\varepsilon^{(0)}_{i1}\psi_1+\varepsilon^{(0)}_{i2}\psi_2$$

$$i=1,2 \tag{2.3a}$$

$$\tilde{f}_{12}[N]=f_{12}-2\varepsilon^{(N-1)}_{12} \qquad \tilde{f}_{21}[N]=f_{21}-2\varepsilon^{(N-1)}_{21} \tag{2.3b}$$

$$\tilde{u}_{11}[N]=u_{11}-2\varepsilon^{(N-1)}_{11,x}+2\varepsilon^{(N-1)}_{12}\varepsilon^{(N-1)}_{21}-f_{12}\varepsilon^{(N-1)}_{21}-f_{21}\varepsilon^{(N-1)}_{12} \tag{2.3c}$$

$$\tilde{u}_{12}[N]=u_{12}+Nf_{12,x}-2\varepsilon^{(N-1)}_{12,x}-2\varepsilon^{(N-2)}_{12}+f_{12}(\varepsilon^{(N-1)}_{11}-\varepsilon^{(N-1)}_{22})+2\varepsilon^{(N-1)}_{12}\varepsilon^{(N-1)}_{22} \tag{2.3d}$$

$$\tilde{u}_{21}[N]=u_{21}+Nf_{21,x}-2\varepsilon^{(N-1)}_{21,x}-2\varepsilon^{(N-2)}_{21}+f_{21}(\varepsilon^{(N-1)}_{22}-\varepsilon^{(N-1)}_{11})+2\varepsilon^{(N-1)}_{21}\varepsilon^{(N-1)}_{11} \tag{2.3e}$$

$$\tilde{u}_{22}[N]=u_{22}-2\varepsilon^{(N-1)}_{22,x}+2\varepsilon^{(N-1)}_{12}\varepsilon^{(N-1)}_{21}-f_{12}\varepsilon^{(N-1)}_{21}-f_{21}\varepsilon^{(N-1)}_{12} \tag{2.3f}$$

where $\varphi^{(j)}=(\varphi^{(j)}_1, \varphi^{(j)}_2)^T$ are solutions of (2.1) with the spectral parameters $\mu^{(j)}$, respectively ($j=1,2,\ldots,2N$); $\Delta[N]$ is the determinant of the $2N\times 2N$ matrix which are obtained from the $(2N+1)\times(2N+1)$ matrix to be written in the numerator of (2.3a) by omitting the first row and column; $\varepsilon^{(K)}_{ij}$ are given by expanding the determinants and comparing the coefficients of $\psi_{j,Kx}(i,j=1,2,\ldots,N-1)$; the indices Kx after the comma denote the Kth-order derivative with respect to x. The formulas for transformed $\tilde{\varphi}^{(j)}_i$ are derived from (2.3a) expanding the wavefunction components:

$$\psi_i(\lambda)=(\lambda-\mu^{(j)})^{-1}(\varphi^{(j)}_i+(\lambda-\mu^{(j)})\partial\varphi^{(j)}_i/\partial\mu^{(j)}\ldots)$$

and taking the limit $\lambda\to\mu^{(j)}$.

Equation (2.2) rewritten for components of the wavefunction coincide with (2.3) for $N=1$. Carrying out the DT of (2.2) on the wavefunctions $\tilde{\varphi}^{(2N+1)}[N]$ and $\tilde{\varphi}^{(2N+2)}[N]$ with the potentials $\tilde{F}[N]$ and $\tilde{U}[N]$ given by (2.3) one obtains (2.3) with $N+1$ instead of N, after some simple determinant algebra. It is seen that all $\varphi^{(j)}$ enter uniformly in equations (2.3).

The existence of different kinds of automorphisms in wavefunction space causes special constraints on the potentials. The automorphism $\psi(\lambda)\leftarrow\psi(\lambda^*)^*$ (the dependence on x is omitted) takes place when the potentials U and F are real. The reduction constraints

$$f_{12}=f_{21}=f \qquad u_{11}=u_{22}=u \qquad u_{12}=u_{21}=v \tag{2.4}$$

give another automorphism $\psi(\lambda)\leftarrow\sigma_1\psi(-\lambda)$ where σ_1 is the Pauli matrix. Both automorphisms hold when $(f,u,v)\in\mathbb{R}$. The Darboux covariance properties for them are summarized in the following lemma.

Lemma 2.2. (i) Let $U,F\in\mathbb{R}$. Then $\tilde{U}[N]$ and $\tilde{F}[N]$ are also real when the following conditions are imposed on the wavefunctions in equation (2.3): $\varphi^{(2L)}=\varphi^{(2L-1)*}$, $\mu^{(2L)}=\mu^{(2L-1)*}$ (or $(\varphi^{(2L-1)}, \varphi^{(2L)})\in\mathbb{R}$ when $(\mu^{(2L-1)}, \mu^{(2L)})\in\mathbb{R}$), $L=1,2,\ldots,N$.

(ii) Let the potentials be of the form (2.4) (i.e. $U^T=U$, $F^T=F$). Then $\tilde{U}[N]^T=\tilde{U}[N]$ and $\tilde{F}[N]^T=\tilde{F}[N]$ when

$$\varphi^{(2L)}=\sigma_1\varphi^{(2L-1)} \qquad \mu^{(2L)}=-\mu^{(2L-1)}. \tag{2.5}$$

The formulas for transformed potentials are

$$\tilde{f}[N]=f-2\varepsilon_2^{(N-1)} \tag{2.6a}$$

$$\tilde{u}[N]=u-2\varepsilon_{1,x}^{(N-1)}+2(\varepsilon_2^{(N-1)}-f)\varepsilon_2^{(N-1)} \tag{2.6b}$$

$$\tilde{v}[N]=v+Nf_x-2\varepsilon_{2,x}^{(N-1)}-2\varepsilon_2^{(N-2)}+2\varepsilon_1^{(N-1)}\varepsilon_2^{(N-1)} \tag{2.6c}$$

where $\varepsilon_1^{(L)}=\varepsilon_{11}^{(L)}=\varepsilon_{22}^{(L)}$, $\varepsilon_2^{(L)}=\varepsilon_{12}^{(L)}=\varepsilon_{21}^{(L)}$.

(iii) Let $(f, u, v)\in\mathbb{R}$. Then the conditions (2.5) added to the conditions: $\varphi^{(2L+1)}=\varphi^{(2L-1)*}$, $\mu^{(2L+1)}=\mu^{(2L-1)*}$ (or $\varphi^{(2L-1)}\in\mathbb{R}$ when $\mu^{(2L-1)}\in\mathbb{R}$ or $\varphi_1^{(2L-1)}=\varphi_2^{(2L-1)*}$ when $\mathrm{i}\mu^{(2L-1)}\in\mathbb{R}$, $\mathrm{i}=\sqrt{-1}$) provide that the transformed potentials possess the same reduction $(\tilde{f}[N], \tilde{u}[N], \tilde{v}[N])\in\mathbb{R}$.

The proof is straightforward.

Remark 2.1. In conditions (ii) of the previous lemma for the case $f=v=0$ one can put $\varphi_1^{(2L-1)}=\varphi^{(2L-1)}$, $\varphi_2^{(2L-1)}=0$ and obtain the formulas of the iterated classical DT for the wavefunctions $\tilde{\psi}_i[N]^C (i=1,2)$ and the potential $\tilde{u}[N]^C$.

3. The bilinear forms and the deep reductions

In this section we consider equation (2.1) with the potentials given by equation (2.4). There are two additional constraints (deep reductions) defined by the corresponding bilinear δ-forms:

$$f=0 \tag{3.1}$$

$$\delta(\psi,\chi)=\psi_{1,x}\chi_1-\psi_1\chi_{1,x}+\psi_{2,x}\chi_2-\psi_2\chi_{2,x}$$

$$v=f_x/2 \tag{3.2}$$

$$\delta(\psi,\chi)=\psi_{1,x}\chi_1-\psi_1\chi_{1,x}-(\psi_{2,x}\chi_2-\psi_2\chi_{2,x})-f(\psi_1\chi_2-\psi_2\chi_1)$$

where $\psi=(\psi_1,\psi_2)^T$, $\chi=(\chi_1,\chi_2)^T$ are solutions of (2.1). The δ-form is independent of x iff the appropriate reduction is imposed and the spectral parameters of the wavefunctions ψ and χ are equal.

Theorem 3.1. When the potentials are of one of the reductions (3.1) or (3.2), equation (2.3*a*) (under the conditions (2.5)) and equations (2.6) constitute the DT which reserve this reduction if $N=2M$, $\mu^{(4L-1)}=\mu^{(4L-3)}$ and 'orthogonality conditions' are imposed on wavefunctions

$$\delta(\varphi^{(4L-1)}, \varphi^{(4L-3)})=0 \qquad L=1,\ldots,M$$

(the δ-forms conform to the reduction constraint).

The 'orthogonality conditions' are then covariant

$$\delta(\tilde{\psi}[N], \tilde{\chi}[N])=0 \text{ if } \delta(\psi,\chi)=0.$$

Proof. The case $M=1$ is checked by direct calculations. Suppose the theorem holds for $M>1$. Performing the DT on the theorem for $N=2$ on the wavefunctions $\tilde{\varphi}^{(4M+i)}[2M]$, $i=1,\ldots,4$ with proper constraints for the reduction cases (3.1) or (3.2) one obtains that the transformed potentials possess the reduction. On the other hand, the formulas obtained by this method are just those for the DT with $(M+1)$ instead of M due to the covariance of the 'orthogonality conditions'.

For real potentials u, v and f the reality of transformed ones is achieved as in the previous section.

Remark 3.1. $\delta(\tilde{\psi}[N]^C, \chi[N]^C)=0$ if $\delta(\psi,\chi)=0$.

4. The Lax pair and solitons of nonlinear equations

The spectral problem equation (2.1) can be considered as the first equation of the Lax pair. Choosing the second one in the form

$$\psi_t=\psi_{3x}+B\psi_x+C\psi \tag{4.1}$$

where

$$\begin{aligned}
B&=\tfrac{3}{2}\operatorname{diag}U+\tfrac{3}{2}F_x+\tfrac{3}{4}F^2\\
C&=\tfrac{3}{2}U_x-\tfrac{3}{4}\operatorname{diag}U_x-\tfrac{3}{4}(f_{12}u_{21}+f_{21}u_{12})I+\tfrac{3}{8}(f_{12,x}f_{21}-f_{12}f_{21,x})\sigma_3\\
&\quad+\tfrac{3}{4}(u_{11}-u_{22})\sigma_3F
\end{aligned}$$

one arrives at the compatibility conditions

$$\begin{aligned}
&F_t-F_{3x}+B_{2x}-3U_{2x}+2C_x+FB_x-\sigma_3B\sigma_3F_x+UB-\sigma_3B\sigma_3U+FC\\
&\qquad-\sigma_3C\sigma_3F=0 &(4.2a)\\
&U_t-U_{3x}+C_{2x}+UC-\sigma_3C\sigma_3U+FC_x-\sigma_3B\sigma_3U_x=0. &(4.2b)
\end{aligned}$$

The automorphisms $\psi(\lambda)\leftarrow\psi(\lambda^*)^*$ and $\psi(\lambda)\leftarrow\sigma_1\psi(-\lambda)$ exist in the space of the solutions of the Lax pair. In the latter case the compatibility conditions (4.2) are

$$\begin{aligned}
&f_t+\tfrac{1}{2}f_{xxx}+\tfrac{3}{2}(uf)_x-\tfrac{3}{4}f_xf^2=0\\
&u_t-\tfrac{1}{4}u_{xxx}-\tfrac{3}{2}u_xu+3v_xv+\tfrac{3}{4}u_xf^2-\tfrac{3}{2}(f_xv)_x=0\\
&v_t+\tfrac{1}{2}v_{xxx}+\tfrac{3}{2}v_xu-\tfrac{3}{4}(vf^2)_x+\tfrac{3}{4}u_{xx}f+\tfrac{3}{2}u_xf_x=0.
\end{aligned}\tag{4.3}$$

Moreover equations (4.3) allow both kinds of deep reduction presented in section 3 (i.e. they are compatible with time evolution). The former reduction ($f=0$) gives rise to the HS system and the latter ($v=f_x/2$) leads to the coupled KdV–MKdV system.

The Lax pair (2.1) and (4.1) and consequently the compatibility conditions are in possession of the same Darboux covariance properties as (2.1) itself. Equations (2.3) constitute the iterated DTs where $\varphi^{(j)}$ are the solutions of this Lax pair. Both kinds of automorphisms being admissible with the Lax pair, Lemma 2.2 is valid. The classical DTs for the KdV equation $\tilde{\psi}[N]^c$ and $\tilde{u}[N]^c$ are also obtained. The bilinear δ-forms are independent of x and t when the spectral parameters of the Lax pair solutions entering in the δ-form are equal and the appropriate constraints are imposed on the potentials. Because of this Theorem 3.1 holds and gives the DTs for the Lax pair (2.1) and (4.1) and its compatibility conditions under the deep reduction constraints.

This way we are able to construct different kinds of solutions. The real potentials can be built by the DTs of Theorem 3.1 when $M=1$; $\mu^{(1)}$ is purely imaginary and $\varphi^{(4)}=\varphi^{(1)*}$ (or $\varphi_2^{(1)}=\varphi_1^{(1)*}$ and $\varphi_2^{(4)}=\varphi_1^{(4)*}$). Beginning from the zero background $f=u=v=0$($\varphi^{(j)i}$ are the sum of the exponents) and imposing the relevant constraints on the wavefunctions (i.e. $\delta(\varphi^{(1)}, \varphi^{(3)})=0$) produces a two-parameter one-soliton solution of the HS system:

$$u=2ln_{xx}\Delta \qquad v=(2+2d^2)^{1/2}\mu_0^2/\Delta \tag{4.4}$$

$$\Delta=\cosh(\mu_0 x-\tfrac{1}{2}\mu_0^3 t)+d\cos(\mu_0 x+\tfrac{1}{2}\mu_0^3 t)$$

or a two-parameter solution for the coupled KdV–MKdV system:

$$f=2(2+2d^2)^{1/2}\Delta_x/\Delta^2 \qquad u=f^2/2+2\ln_{xx}\Delta$$

where $\mu^{(1)}=i\mu_0^2/2$, μ_0 and the configuration parameter d are real constants. For small $|d|$ this solution is a smooth function. However, when $|d|>1$ poles appear and the number of poles increases with $|d|$. When $M=2$ the real six-parameter soliton solutions are obtained putting $\mu^{(5)}=\mu^{(1)*}$ ($\mathrm{Re}\,\mu^{(1)}\neq 0$) and $\varphi^{(5)}=\varphi^{(1)*}$. Alternatively, if $\mu^{(1)}$ and $\mu^{(5)}$ ($\mu^{(1)}\neq\mu^{(5)}$) are purely imaginary the two-soliton solution of the two-parameter family is produced. The interaction does not change the configuration parameters and only changes the relative location and phase shifts.

One can perform DTs for deep reductions not only on zero background but on any solution of the KdV equation. Remark 3.1 is useful in the construction of solutions which describe the interaction of the KdV and deep reduction (e.g. HS system) solitons.

Setting $d=0$ in (4.4) one obtains the well known single-parameter soliton of the NS system. While this solution is obtained from the two-soliton solution in the Sato theory, the two-parameter soliton is extracted from the four-soliton solution (from the eight-soliton one for the six-parameter solution) by means of more complicated constraints than given in [4]. However, we have not succeeded in proving general formulas which contain the KdV, one- and two- (or six-) parameter soliton solutions for both deep reductions in the framework of the Sato theory. This is a matter of further study.

5. The Yajima–Oikawa system

In this section we briefly consider the Lax pair which includes the Zakharov–Shabat spectral problem of the third order

$$\psi_x+\lambda J\psi=U\psi \tag{5.1}$$

and the linear evolution equation of the form

$$\psi_t=\lambda^2 A\psi+\lambda B\psi+C\psi \tag{5.2}$$

where $\psi=(\psi_1, \psi_2, \psi_3)^T$ is a solution of the Lax pair with the spectral parameter λ, $J=\mathrm{diag}(1, 0, -1)$, $A=\mathrm{diag}(\mathrm{i}, 0, \mathrm{i})$,

$$U=\{u_{ij}, u_{ii}=0\}$$

$$B=\mathrm{i}\begin{pmatrix} 0 & -u_{12} & 0 \\ -u_{21} & 0 & u_{23} \\ 0 & u_{32} & 0 \end{pmatrix}$$

$$C=\mathrm{i}\begin{pmatrix} -u_{12}u_{21} & u_{12,x}+u_{13}u_{32} & u_{12}u_{23} \\ -u_{21,x}+u_{23}u_{31} & u_{12}u_{21}+u_{32}u_{23} & -u_{23,x}+u_{21}u_{13} \\ u_{32}u_{21} & u_{32,x}+u_{12}u_{31} & -u_{23}u_{32} \end{pmatrix}.$$

The compatibility condition of equations (5.1) and (5.2) is

$$U_t=C_x+CU-UC. \tag{5.3}$$

There are at least two reduction constraints on the potential U (presented below) which are admissible with time evolution.

The bilinear δ-form

$$\delta(\psi, \chi)=\psi_1\chi_3^*+\psi_2\chi_2^*+\psi_3\chi_1^*$$

on the solutions ψ and χ of equations (5.1) and (5.2) with spectral parameters λ and λ_1 is independent of both x and t when $\lambda_1=\lambda^*$ and the following constraints are imposed:

$$u_{23}=-u_{12}^* \qquad u_{21}=-u_{32}^* \qquad u_{13}+u_{13}^*=0 \qquad u_{31}+u_{31}^*=0. \tag{5.4}$$

The reduction constraints

$$u_{21}=u_{32}=0 \qquad u_{31}=\mathrm{i} \tag{5.5}$$

lead to the symmetry property: if $\psi=(-i(\psi_{3,x}-\lambda\psi_3), \psi_2, \psi_3)^T$ is the solution with spectral parameter λ, then $\chi=(-\mathrm{i}(\psi_{3,x}+\lambda\psi_3), \psi_2, \psi_3)$ is the solution with spectral parameter $(-\lambda)$.

The product of these constraints (deep reduction) are also admissible with time evolution. The compatibility condition turns then into the Yajima–Oikawa (YO) system [8]:

$$u_{12,t}=\mathrm{i}u_{12,xx}+u_{12}u_{13} \tag{5.6a}$$

$$u_{13,t}=-2\mathrm{i}(|u_{12}|^2)_x. \tag{5.6b}$$

Equations (5.1) and (5.2) with these reductions are written in terms of components ψ_2 ψ_3 only. The first is

$$\psi_{2,x}=-u_{12}^*\psi_3 \tag{5.7a}$$

$$\psi_{3,xx}=\lambda^2\psi_3+\mathrm{i}u_{13}\psi_3+\mathrm{i}u_{12}\psi_2. \tag{5.7b}$$

The Lax pair and hence the compatibility condition are covariant with respect to the DT [2]:

$$\tilde{\psi}=\lambda\psi+\varepsilon\psi \qquad \varepsilon=-\varphi\Lambda\varphi^{-1} \tag{5.8a}$$

$$\tilde{U}=U+J\varepsilon-\varepsilon J \tag{5.8b}$$

where the matrix φ is $\varphi=(\varphi^{(1)}, \varphi^{(2)}, \varphi^{(3)})$, $\varphi^{(i)}=(\varphi_1^{(i)}, \varphi_2^{(i)}, \varphi_3^{(i)})^T$ $(i=1, 2, 3)$ are the solutions of (5.1) and (5.2) with spectral parameters $\mu^{(i)}$, $\Lambda=\mathrm{diag}\{\mu^{(1)}, \mu^{(2)}, \mu^{(3)}\}$.

Similarly to section 2 one arrives at determinant formulas rewriting (5.8) for the components of the wavefunction ψ. The N-times iteration of this DT leads to cumbersome formulas in the determinants of the orders $(3N+1)$ and $3N$ which include different wavefunctions $\varphi^{(i)}$ $(i=1,2,\ldots,3N)$ in a uniform manner.

Let the potential U possesses the reduction constraints (5.4). Imposing the conditions $\mu^{(2)}=\mu^{(3)}=\mu^{(1)*}$, $\delta(\varphi^{(1)},\varphi^{(2)})=\delta(\varphi^{(1)},\varphi^{(3)})=0$ provides that the transformed potential $\tilde{U}$ satisfies the same reduction. Then equations (5.8) take the form

$$\tilde{\psi}_i=\begin{vmatrix}\psi_i & \delta^s(\psi,\varphi)\\ \varphi_i & \delta^s(\varphi,\varphi)\end{vmatrix}\Big/\delta^s(\varphi,\varphi)=\psi_i+d_i\delta^s(\psi,\varphi) \tag{5.9a}$$

$$\tilde{u}_{12}=u_{12}+d_1\varphi_2^* \qquad \tilde{u}_{32}=u_{32}-d_3\varphi_2^* \tag{5.9b}$$

$$\tilde{u}_{13}=u_{13}+2d_1\varphi_1^* \qquad \tilde{u}_{31}=u_{31}-2d_3\varphi_3^* \tag{5.9c}$$

where $\varphi=\varphi^{(1)}$, $\mu=\mu^{(1)}$, $\delta^s(\psi,\varphi)=\delta(\psi,\varphi)/(\lambda-\mu^*)$, $d_i=-\varphi_i/\delta^s(\varphi,\varphi)$, $i=1,2,3$. The 'orthogonality conditions' for δ-forms are covariant: $\delta(\tilde{\psi}[1],\tilde{\chi}[1])=0$ if $\delta(\psi,\chi)=0$.

Equations (5.9) constitute the DT for (5.1), (5.2) and (5.3) under the reduction (5.4). The iterations of this DT give

$$\tilde{\psi}_i[N]=\begin{vmatrix}\psi_i & \delta^s(\psi,\varphi^{(1)}) & \ldots & \delta^s(\psi,\varphi^{(N)})\\ \varphi_i^{(1)} & \delta^s(\varphi^{(1)},\phi^{(1)} & \ldots & \delta^s(\varphi^{(1)},\phi^{(N)}\\ \vdots & \vdots & \vdots & \vdots\\ \varphi_i^{(N)} & \delta^s(\varphi^{(N)},\varphi^{(1)}) & \ldots & \delta^s(\varphi^{(N)},\varphi^{(N)})\end{vmatrix}\Big/\Delta[N]$$

$$=\psi_i+d_i^{(j)}\delta^s(\psi,\varphi^{(j)}) \qquad i=1,2,3 \tag{5.10a}$$

$$\tilde{u}_{12}[N]=u_{12}+d_1^{(j)}\varphi_2^{(j)*} \qquad \tilde{u}_{32}[N]=u_{32}-d_3^{(j)}\varphi_2^{(j)*} \tag{5.10b}$$

$$\tilde{u}_{13}[N]=u_{13}+2d_1^{(j)}\varphi_1^{(j)*} \qquad \tilde{u}_{31}[N]=u_{31}-2d_3^{(j)}\varphi_3^{(j)*} \tag{5.10c}$$

where $\Delta[N]=\det\{\delta^s(\varphi^{(i)},\varphi^{(j)})\}$; $\varphi^{(i)}$ $(i=1,2,\ldots,N)$ are the solutions with the spectral parameters $\mu^{(i)}$ of equations (5.1) and (5.2) (we use the symbol $\varphi^{(i)}$ as in (5.8) for simplicity, but here any constraints are not yet imposed on them); $d_i^{(j)}$ are defined by expansion of the determinants; summation over repeated indices in (5.10) is assumed. The 'orthogonality conditions' covariance holds: $\delta(\tilde{\psi}[N],\tilde{\chi}[N])=0$ if $\delta(\psi,\chi)=0$.

The Equation (5.10) are a specialization of the N-times iteration of (5.8). The complete sets of 'orthogonal' wavefunctions (in the sense of the δ-forms) being used, the final formulas (5.10) do not depend on them. That (5.10) are the DT for the reduction is also proved by direct substitution in the Lax pair.

Let both reduction constraints be imposed on the potential U. Putting $N=2M$ and

$$\mu^{(M+i)}=-\mu^{(i)} \qquad \varphi^{(M+i)}=(-i(\varphi_{3,x}^{(i)}+\mu^{(i)}\varphi_3),\varphi_2^{(i)},\varphi_3^{(i)})^T \tag{5.11}$$

(i.e. $\varphi^{(i)}$ and $\varphi^{(M+i)}$ are connected by the symmetry property) one obtains that the DTs given by (5.10) with the additional conditions (5.11) maintain the reduction constraints of (5.4) and (5.5).

Since equations (5.10) are the iteration formulas it is sufficient to check this statement for $M=1$ and to prove the 'heredity theorem': $\tilde{\psi}[2]$ and $\tilde{\chi}[2]$ are connected by the symmetry property if ψ and χ are connected.

Excluding the components ψ_1 and $\varphi_1^{(i)}$ of the wavefunctions from (5.10) the DT for the Lax pair in the form of (5.7) and for the YO system are written

$$\tilde{\psi}_i[2M] = \begin{vmatrix} \psi_i & \Delta(\psi, \varphi^{(1)}) & \dots & \Delta(\psi, \varphi^{(M)}) \\ \varphi_i^{(1)} & \Delta(\varphi^{(1)}, \phi^{(1)}) & \dots & \Delta(\varphi^{(1)}, \phi^{(M)}) \\ \vdots & \vdots & \vdots & \vdots \\ \varphi_i^{(M)} & \Delta(\varphi^{(M)}, \varphi^{(1)}) & \dots & \Delta(\varphi^{(N)}, \varphi^{(M)}) \end{vmatrix} \Bigg/ \Delta[2M]$$

$$= \psi_i + D_i^{(j)} \Delta(\psi, \varphi^{(j)}) \qquad i = 1, 2, 3 \tag{5.12a}$$

$$\tilde{u}_{13}[2M] = u_{13} - (2i) \ln_{xx} \Delta[2M] \tag{5.12b}$$

$$\tilde{u}_{12}[2M] = u_{12} - D_3^{(j)} \varphi_2^{(j)*} \tag{5.12c}$$

where

$$\Delta[2M] = \det\{\Delta(\varphi^{(i)}, \varphi^{(j)})\}$$

$$\Delta(\varphi^{(i)}, \varphi^{(j)}) = (\varphi_{3,x}^{(i)} \varphi_3^{(j)*} - \varphi_3^{(i)} \varphi_{3,x}^{(j)*} + i\varphi_2^{(i)} \varphi_2^{(i)}) / (\mu^{(i)2} - \mu^{(j)*2}).$$

The final formulas are proved immediately.

Performing the DT of (5.12) for $M = 1$ with the wavefunction $\varphi^{(1)}$ the solution of the Lax pair (5.1) and (5.2) on the zero background ($u_{12} = u_{13} = 0$ but $u_{31} = i$), gives the four-parameter solitons which are singular except for the two-parameter (velocity and frequency) soliton solutions [8].

6. Conclusion

In this paper we have presented a method of constructing solutions for the reductions of nonlinear evolution equations. New soliton solutions have been revealed for the Hirota–Satsuma and coupled KdV–MKdV systems. Although the collapsing states are known the noteworthy feature of these new solutions is the dependence on the free parameter.

The technique suggested here can be extended to other deep reductions of the Mikhailov type [1] (e.g. the Boullough–Dodd–Zhiber–Shabat equation). Other kinds of deep reductions including the Maxwell–Bloch equation, complex MKdV, Sawada–Kotera, Kaup–Kupershmidt equations can also be treated, with some modifications. The reduction constraints define the symmetry properties in wavefunction space which turns out to be useful in extracting the proper potentials form the general formulas of the Darboux transformations.

We hope that results concerning these equations will be published elsewhere.

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References

[1] Mikhailov A V 1981 The reduction problem and the inverse scattering method *Physica* **D3** 73–117
[2] Matveev V B and Salle M A 1991 *Darboux Transformations and Solitons* (Berlin: Springer)
[3] Hirota R and Satsuma J 1981 Soliton solution of a coupled KdV equation *Phys. Lett.* **85A** 407–9

[4] Satsuma J and Hirota R 1982 Coupled KdV equations is a special case of the KP hierarchy *J. Phys. Soc. Japan* **51** 3390–7
[5] Dodd R and Fordy A 1982 On the integrability of a system of coupled KdV equations *Phys. Lett.* **89A** 168–71
[6] Levi D 1982 A hierarchy of coupled Korteweg-de Vries equations *Phys. Lett.* **95A** 7–10
[7] Leble S B and Ustinov N V 1990 KdV–MKdV systems and Darboux transforms in 1+1 and 2+1 dimensions *Preprint* 1/3 10.90, Kaliningrad State Univ. *J. Math. Phys.* **34**
[8] Yajima N and Oikawa M 1976 Formation and interaction of sonic-Langmuir solitons *Prog. Theor. Phys.* **56** 1719–39
Ma Y-C 1978 The complete solution of the long-wave–short-wave resonance equation *Stud. Appl. Math.* **59** 201–21